

Operator-Like Wavelet Bases of $L_2(\mathbb{R}^d)$ *

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Abstract

The connections between derivative operators and wavelets are well known. Here we generalize the concept by constructing multiresolution approximations and wavelet basis functions that act like differential Fourier multiplier operators. This construction follows from a stochastic model: signals are tempered distributions such that the application of a whitening (differential) operator results in a realization of a sparse white noise. Using wavelets constructed from these operators, the sparsity of the white noise can be inherited by the wavelet coefficients. In this paper, we specify such wavelets in full generality and determine their properties in terms of the underlying operator.

1 Introduction

In the past few decades, a variety of wavelets that provide a complete and stable multiscale representation of $L_2(\mathbb{R}^d)$ have been developed. The wavelet decomposition is very efficient from a computational point of view, due to the fast filtering algorithm. Underlying this approximation procedure is a differential operator, so our purpose in this paper is to construct wavelets that behave like a given differential Fourier multiplier operator L , which can be more general than a pure derivative. In our approach, the multiresolution

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spaces are characterized by generalized B-splines associated with the operator, and we show that, in a certain sense, the wavelet inherits properties of the operator. Importantly, the operator-like wavelet can be constructed directly from the operator, bypassing the scaling function space. What makes the approach even more attractive is that, at each scale, the wavelet space is generated by the shifts of a single function. Our work provides a generalization of some known constructions including: cardinal spline wavelets [5], elliptic wavelets [15], polyharmonic spline wavelets [19, 20], Wirtinger-Laplace operator-like wavelets [21], and exponential-spline wavelets [13].

In applications, it has been observed that many signals are well represented by a relatively small number of wavelet coefficients. Interestingly, the model that motivates our wavelet construction explains the origin of this sparsity. The context is that of sparse stochastic processes, which are defined by a stochastic differential equation driven by a (non-Gaussian) white noise. Explicitly, the model states that $Ls = w$ where the signal s is a tempered distribution, L is a shift-invariant Fourier multiplier operator, and w is a sparse white noise [18]. The wavelets we construct are designed to act like the operator L so that the wavelet coefficients are determined by a generalized B-spline analysis of w . In particular, we define an interpolating spline ϕ , corresponding to L^*L , from which we derive the wavelets $\psi = L^*\phi$. Then the wavelet coefficients are formally computed by the L_2 inner product

$$\langle s, \psi \rangle = \langle s, L^*\phi \rangle = \langle Ls, \phi \rangle = \langle w, \phi \rangle.$$

Sparsity of w combined with localization of the B-splines results in sparse wavelet coefficients. This model is relevant in medical imaging applications, where good performance has been observed in approximating functional magnetic resonance imaging and positron emission tomography data using operator-like wavelets that are tuned to the hemodynamic or pharmacokinetic response of the system [12, 22].

Our construction falls under the general setting of pre-wavelets, which are comprehensively discussed by de Boor, DeVore, and Ron [7]. Two distinguishing properties of our approach are its operator-based nature and the fact that it is non-stationary. Related constructions have been developed for wavelets based on radial basis functions [4, 6, 17]. In fact [6] also takes an operator approach; however, the authors were focused on wavelets defined on arbitrarily spaced points.

This paper is organized as follows. In Section 2, we formally define the

class of admissible operators and the lattices on which our wavelets are defined. In Section 3, we construct the non-stationary multiresolution analysis (MRA) that corresponds to a given operator L and derive approximation rates for functions lying in Sobolev-type spaces. Then, in Section 4, we introduce the operator-like wavelets and study their properties; in particular, we derive conditions on L that guarantee that our choice of wavelet yields a stable basis at each scale. Under an additional constraint on L , we use this result to define Riesz bases of $L_2(\mathbb{R}^d)$. In Section 5, we prove a decorrelation property for families of related wavelets. Finally, we conclude in Section 6 with some examples of the wavelets provided by our construction.

2 Preliminaries

The primary objects of study in this paper are differential Fourier multiplier operators and their derived wavelets. The operators we consider are shift-invariant operators L that act on the class of square integrable functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$, which will be denoted by $L_2(\mathbb{R}^d)$. The action of such a Fourier multiplier operator is defined by its symbol \widehat{L} in the Fourier domain, with

$$Lf = \left(\widehat{L}\widehat{f} \right)^\vee,$$

where the symbol \widehat{L} is a measurable function. Here \widehat{f} denotes the Fourier transform of f

$$\widehat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{i\boldsymbol{x} \cdot \boldsymbol{\omega}} d\boldsymbol{x},$$

and g^\vee denotes the inverse Fourier transform of g . Pointwise values of \widehat{L} will be required for some of our analysis, so we restrict the class of symbols by requiring continuity almost everywhere. Additionally, we would like to have a well-defined inverse of the symbol, so \widehat{L} should not be zero on a set of positive measure. To be precise, we define the class of admissible operators as follows.

Definition 2.1. Let L be a Fourier multiplier operator, then L is admissible if its symbol \widehat{L} is a ratio of continuous functions f/g satisfying

1. The set of zeros of fg has Lebesgue measure zero;

2. The zero sets of f and g are disjoint.

Notice that each such operator defines a subspace of L_2 , consisting of functions whose derivatives are also square integrable, and our approximation results will focus on these spaces.

Definition 2.2. An admissible operator defines a Sobolev-type subspace of $L_2(\mathbb{R}^d)$:

$$W_2^L(\mathbb{R}^d) = \left\{ f \in L_2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\omega})|^2 (1 + |\widehat{L}(\boldsymbol{\omega})|^2) d\boldsymbol{\omega} < \infty \right\}.$$

Having defined the class of admissible operators, we must consider the lattices on which the multiresolution spaces will be defined. It is important to use lattices which are nested, so we consider those defined by an expansive integer matrix. Specifically, an integer matrix \mathbf{A} , whose eigenvalues are all larger than 1 in absolute value, defines a sequence of lattices

$$\mathbf{A}^j \mathbb{Z}^d = \{\mathbf{A}^j \mathbf{k} : \mathbf{k} \in \mathbb{Z}^d\}$$

indexed by an integer j . Here, we state several results about these lattices that will be used in this paper, most of which can be found in [11]. First, we know that $\mathbf{A}^j \mathbb{Z}^d$ can be decomposed into a finite union of disjoint copies of $\mathbf{A}^{j+1} \mathbb{Z}^d$; there are $|\det(\mathbf{A})|$ vectors $\{\mathbf{e}_l\}_{l=0}^{|\det(\mathbf{A})|-1}$ such that

$$\bigcup_l (\mathbf{A}^j \mathbf{e}_l + \mathbf{A}^{j+1} \mathbb{Z}^d) = \mathbf{A}^j \mathbb{Z}^d,$$

and using this notation, our convention will be to set $\mathbf{e}_0 = \mathbf{0}$.

There are also several important properties that arise when using Fourier techniques on more general lattices. First, a lattice in the spatial domain corresponds to a lattice in the Fourier domain known as its dual lattice, and the dual lattice of $\mathbf{A}^j \mathbb{Z}^d$ is given by $2\pi(\mathbf{A}^T)^{-j} \mathbb{Z}^d$. Also relevant is the notion of a fundamental domain, which for $\mathbf{A}^j \mathbb{Z}^d$ is a measurable set Ω_j satisfying

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \chi_{\Omega_j}(\mathbf{x} + \mathbf{A}^j \mathbf{k}) = 1$$

for all \mathbf{x} . Finally, just as in the dyadic case, we can define a discrete Fourier transform (DFT) corresponding to \mathbf{A} in terms of the functions which are periodic with respect to the lattice. The matrix for this transform is given by

$$\mathbf{F} = (e^{-i\mathbf{k}\cdot\mathbf{l}})_{\mathbf{k},\mathbf{l}},$$

and the corresponding inverse DFT matrix is

$$\mathbf{F}^{-1} = \frac{1}{|\det((\mathbf{A}^T)^{-1})|} (e^{i\mathbf{k}\cdot\mathbf{l}})_{\mathbf{l},\mathbf{k}},$$

where \mathbf{k} and \mathbf{l} range over the cosets of $2\pi(\mathbf{A}^T)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ and $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, respectively. The inverse DFT matrix will play a role in deriving the Riesz basis condition on the wavelet spaces. We summarize the required properties in the following proposition.

Proposition 2.3. *A lattice generated by an expansive integer matrix \mathbf{A} defines a discrete Fourier transform. This transform has a matrix representation which is invertible and has nonzero entries.*

3 Multiresolution Analysis

The multiresolution framework for wavelet construction was presented by Mallat in the late 1980s [14]. In the following years, the notion of pre-wavelets was developed, and a more general notion of multiresolution was adopted. We consider this more general setting in order to allow for a wider variety of admissible operators.

Definition 3.1. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed linear subspaces of $L_2(\mathbb{R}^d)$ forms a non-stationary multiresolution analysis if

1. $V_{j+1} \subseteq V_j$;
2. $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R}^d)$ and $\bigcap_{j\in\mathbb{Z}} V_j$ is at most one-dimensional;
3. $f \in V_j$ if and only if $f(\cdot - \mathbf{A}^j\mathbf{k}) \in V_j$ for all $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, where \mathbf{A} is an expansive integer matrix;
4. For each $j \in \mathbb{Z}$, there is an element $\varphi_j \in V_j$ such that the collection of translates $\{\varphi_j(\cdot - \mathbf{A}^j\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^d\}$ is a Riesz basis of V_j , i.e., there are constants $0 < A_j \leq B_j < \infty$ such that

$$A_j \|c\|_{\ell_2}^2 \leq \left\| \sum_{\mathbf{k} \in \mathbb{Z}^d} c[\mathbf{k}] \varphi_j(\cdot - \mathbf{A}^j\mathbf{k}) \right\|_{L_2(\mathbb{R}^d)}^2 \leq B_j \|c\|_{\ell_2}^2.$$

In order to produce non-stationary MRAs, we require additional properties on the admissible operators. Together with a dilation matrix, the operator should admit generalized B-splines that satisfy decay and stability properties.

As motivation for our definition, let us consider the one-dimensional example where L is defined by

$$Lf(t) = \frac{df}{dt}(t) - \alpha f(t).$$

In this case, a Green's function for L is $\rho(t) = e^{\alpha t}H(t)$, where H is the Heaviside function. In order to produce Riesz bases for the scaling matrix $\mathbf{A} = (2)$, we introduce the localization operator $L_{d,j}$ defined by $L_{d,j}f = f - e^{2^j\alpha}f(\cdot - 2^j)$. The generalized exponential-spline $\varphi_j := L_{d,j}\rho$ will then be a compactly supported function whose shifts $\varphi_j(\cdot - 2^j)$ form a Riesz basis. Notice that, in the Fourier domain, φ_j is given by $\widehat{L}^{-1}(\omega)\widehat{L}_{d,j}(\omega)$ which is

$$\widehat{\varphi}_j(\omega) = \frac{1 - e^{2^j(\alpha - i\omega)}}{i\omega - \alpha}.$$

In this form one can verify the equivalent Riesz basis condition

$$0 < A_j \leq \sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\cdot - \pi 2^{-j+1}\mathbf{k})|^2 \leq B_j < \infty.$$

In fact, based on the symbol of L , we could have worked entirely in the Fourier domain to determine appropriate periodic functions $\widehat{L}_{d,j}$. With this example in mind, we make the following definition.

Definition 3.2. We say that an operator L and an integer matrix \mathbf{D} are a spline-admissible pair of order $r > d/2$ if the following conditions are satisfied:

1. L is an admissible Fourier multiplier operator;
2. $\mathbf{D} = a\mathbf{R}$ with \mathbf{R} an orthogonal matrix and $a > 1$;
3. For some constant $c_L > 0$ and all positive h , we have

$$\left\| \left(1 + |\widehat{L}(\boldsymbol{\omega}/h)|^2 \right)^{-1/2} \right\|_{L_\infty(\mathbb{R}^d \setminus \Omega)} \leq c_L h^r,$$

where Ω is the domain $[-\pi, \pi]^d$;

4. For every $j \in \mathbb{Z}$, there exists a periodic function $\widehat{L}_{d,j}$ such that $\widehat{\varphi}_j(\boldsymbol{\omega}) := \widehat{L}_{d,j}(\boldsymbol{\omega})\widehat{L}(\boldsymbol{\omega})^{-1}$ satisfies the Riesz basis condition

$$0 < A_j \leq \sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2 \leq B_j < \infty,$$

for some A_j and B_j in \mathbb{R} . Here we require the periodic functions $\widehat{L}_{d,j}$ to be of the form $\sum_{\mathbf{k} \in \mathbb{Z}^d} p_j[\mathbf{k}]e^{i\boldsymbol{\omega} \cdot \mathbf{D}^j \mathbf{k}}$ for some $p \in \ell_1(\mathbb{Z}^d)$.

It can be verified that the last condition implies that the function $\widehat{\varphi}_j$ is in $L_2(\mathbb{R}^d)$. We call its inverse Fourier transform $\varphi_j := (\widehat{\varphi}_j)^\vee$ a generalized B-spline for L . Also note that, in this definition, we have restricted the scaling matrices to be multiples of orthogonal matrices. The lattices generated by these matrices have some additional nice properties. For example, the lattices generated by \mathbf{D} and \mathbf{D}^T are the same, and the lattices $\mathbf{D}^j \mathbb{Z}^d$ scale uniformly in j . Also, for such matrices, there are only finitely many possible lattices generated by powers of \mathbf{D} ; i.e., there always exists a positive integer n for which $\mathbf{D}^n = a^n \mathbf{I}$.

Proposition 3.3. *Given a spline-admissible pair L and \mathbf{D} , the spaces*

$$V_j = \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^d} c[\mathbf{k}] \varphi_j(\cdot - \mathbf{D}^j \mathbf{k}) : c \in \ell_2(\mathbb{Z}^d) \right\}$$

form a non-stationary MRA.

Proof. The fact that $V_{j+1} \subseteq V_j$ follows from the definition of \mathbf{D} and the Riesz basis conditions on the B-splines. The density property is a result of the admissibility of L , the Riesz basis condition, and the inclusion relation $V_{j+1} \subseteq V_j$, cf. [7, Theorem 4.3]. In addition, the conditions imposed on L imply that the dimension of the intersection of the spaces in our MRA is at most one, cf. [7, Theorem 4.9]. Lastly, Properties 3 and 4 follow directly from the definitions. \square

Condition 4 in the definition of spline-admissibility is the most challenging to verify; however, there are many operators for which this condition is satisfied. For instance, it holds for any one-dimensional, constant-coefficient differential operator. In addition, it holds in higher dimensions for the Matérn operators characterized by $\widehat{L}(\boldsymbol{\omega}) = (1 + |\boldsymbol{\omega}|^2)^{\nu/2}$, as they require no localization. In Section 6, we shall give a less trivial example and show how this

Riesz basis property can be verified. As a final point, note that if one is only interested in analyzing fine-scale spaces, Condition 4 need only be satisfied for j smaller than a fixed integer j_0 , but in this case, it is necessary to include the space V_{j_0} in the wavelet decomposition.

We shall close this section by determining approximation rates for the spaces $\{V_j\}_{j \in \mathbb{Z}}$, in terms of L and the density of the lattice generated by \mathbf{D}^j . In order to state this result, we must define the spline interpolants for the operator L^*L , whose symbol is $|\widehat{L}|^2$. The spline admissibility of this operator is the subject of the next proposition.

Proposition 3.4. *If L is spline admissible of order $r > d/2$, then L^*L is spline admissible of order $2r > d$.*

Proof. Let L be a spline admissible operator of order $r > d/2$. First notice that L^*L is an admissible Fourier multiplier operator. Also, we can see that L^*L satisfies Condition 3 of Definition 3.2 with r replaced by $2r$. Therefore spline admissibility will follow if we can exhibit B-splines for L^*L that satisfy the Riesz basis condition, where the integer dilation matrix is the same as for L . To this end, we claim that $\widehat{\varphi}_j := |\widehat{\varphi}_j|^2$ defines the Fourier transform of a B-spline for L^*L .

An upper Riesz bound for $\widehat{\varphi}_j$ can be found by using the norm inequality between ℓ_1 and ℓ_2 :

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^4 \leq \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2 \right)^2 \leq B_j^2.$$

To verify the lower Riesz bound for $\widehat{\varphi}_j$, we note that the decay condition of spline admissibility implies that for $R > 0$ sufficiently large, we have

$$\left| \widehat{L} \left(\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} R \right) \right| \leq CR^{-r}.$$

Now for any $M > 0$ sufficiently large, we can bound

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2$$

above by using the decay estimate on L^{-1} :

$$\sum_{|\mathbf{k}| \leq M} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2 + C \left| \widehat{L}_{d,j}(\boldsymbol{\omega}) \right| M^{d-2r} |\det(\mathbf{D})|^{2jr/d}.$$

Therefore

$$\sum_{|\mathbf{k}| \leq M} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2 \geq A_j - C \left| \widehat{L}_{d,j}(\boldsymbol{\omega}) \right| M^{d-2r} |\det(\mathbf{D})|^{2jr/d},$$

and

$$\begin{aligned} \sum_{|\mathbf{k}| \leq M} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^4 &\geq CM^{-d} \left(\sum_{|\mathbf{k}| \leq M} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2 \right)^2 \\ &\geq CM^{-d} \left(A_j - C \left| \widehat{L}_{d,j}(\boldsymbol{\omega}) \right| M^{d-2r} |\det(\mathbf{D})|^{2jr/d} \right)^2 \end{aligned}$$

Due to the fact that $2r > d$, we can always choose M to make the right hand side positive, and this establishes a lower Riesz bound for $\widetilde{\varphi}_j$. \square

The Riesz basis property of the B-splines for L^*L imply that the L^*L -spline interpolants $\phi_j(\mathbf{x})$, given by

$$\widehat{\phi}_j(\boldsymbol{\omega}) = |\det(\mathbf{D})|^j \frac{|\widehat{\varphi}_j(\boldsymbol{\omega})|^2}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2} \quad (1)$$

are well-defined and also generate Riesz bases. Importantly, $\phi_j \in W_2^L$ does not depend on the specific choice of the localization operator, as we can see from

$$\begin{aligned} \widehat{\phi}_j(\boldsymbol{\omega}) &= |\det(\mathbf{D})|^j \frac{\left| \widehat{L}_{d,j}(\boldsymbol{\omega}) \right|^2 \left| \widehat{L}(\boldsymbol{\omega}) \right|^{-2}}{\left| \widehat{L}_{d,j}(\boldsymbol{\omega}) \right|^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \left| \widehat{L}(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k}) \right|^{-2}} \\ &= |\det(\mathbf{D})|^j \frac{1}{1 + \left| \widehat{L}(\boldsymbol{\omega}) \right|^2 \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}} \left| \widehat{L}(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k}) \right|^{-2}}. \end{aligned}$$

These L*L-spline interpolants play a key role in our wavelet construction, which we describe in the next section; however, for our approximation result, we are more interested in the related functions

$$m_j(\boldsymbol{\omega}) = \frac{\left| \widehat{L}(\boldsymbol{\omega}) \right|^{-2}}{\sum_{\mathbf{k} \in \mathbb{Z}^d} \left| \widehat{L}(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k}) \right|^{-2}}, \quad (2)$$

which will also be needed for the decorrelation result in Theorem 5.4.

In order to bound the error of approximation from the spaces V_j , we apply the techniques developed in [8]. In that paper, the authors derive a characterization of certain potential spaces in terms of approximation by closed, shift-invariant subspaces of $L_2(\mathbb{R}^d)$. The same techniques can be applied in our situation, with a few modifications to account for smoothness being determined by different operator norms.

The error in approximating a function $f \in L_2(\mathbb{R}^d)$ by a closed function space X will be denoted by

$$E(f, X) = \min_{s \in X} \|f - s\|_{L_2(\mathbb{R}^d)},$$

and the approximation rate will be given in terms of the density of the lattice in \mathbb{R}^d . The lattice determined by \mathbf{D}^j has density proportional to $|\det(\mathbf{D})|^{j/d}$, so we say that V_j provides approximation order \tilde{r} if for every $f \in W_2^L(\mathbb{R}^d)$

$$E(f, V_j) \leq c |\det(\mathbf{D})|^{j\tilde{r}/d} \|f\|_{W_2^L(\mathbb{R}^d)}.$$

The rate of approximation is then determined by the following theorem.

Theorem 3.5. *For a spline-admissible pair L and \mathbf{D} of order $r > d/2$, the multiresolution spaces V_j provide approximation order $\tilde{r} \leq r$ if*

$$|\det(\mathbf{D}^T)|^{-2j\tilde{r}/d} \frac{1 - m_j(\boldsymbol{\omega})}{1 + \left| \widehat{L}(\boldsymbol{\omega}) \right|^2}$$

is bounded, independently of j , in $L_\infty((\mathbf{D}^T)^{-j}\Omega)$, where $\Omega = [-\pi, \pi]^d$.

Proof. This result is a consequence of [8, Theorem 4.3]. To show this let us introduce the notation $f_j(\cdot) = f(\mathbf{D}^j \cdot)$, which implies that $\widehat{f}_j = |\det(\mathbf{D})|^{-j} \widehat{f} \circ (\mathbf{D}^T)^{-j}$. Notice that the spaces V_j are scaled copies of the integer shift-invariant spaces

$$\begin{aligned}
V_j^j &:= \{s(\mathbf{D}^j \cdot) : s \in V_j\} \\
&= \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^d} c[\mathbf{k}] \varphi_j(\mathbf{D}^j(\cdot - \mathbf{k})) : c \in \ell_2(\mathbb{Z}^d) \right\}.
\end{aligned}$$

We then can write the error of approximating a function $f \in W_2^L(\mathbb{R}^d)$ from V_j in terms of approximation by \widehat{V}_j^j as

$$\begin{aligned}
E(f, V_j) &= |\det(\mathbf{D})|^{j/2} E(f_j, V_j^j) \\
&= (2\pi)^{-d/2} |\det(\mathbf{D})|^{j/2} E(\widehat{f}_j, \widehat{V}_j^j),
\end{aligned}$$

where \widehat{V}_j^j is composed of the Fourier transforms of functions in V_j^j . Separating this last term, we have

$$E(f, V_j) \leq (2\pi)^{-d/2} |\det(\mathbf{D})|^{j/2} \left(E(\widehat{f}_j \chi_\Omega, \widehat{V}_j^j) + \left\| (1 - \chi_\Omega) \widehat{f}_j \right\|_2 \right), \quad (3)$$

where χ_Ω is the characteristic function of the set Ω . We are now left with bounding both terms on the right-hand side of (3). First, we have

$$\begin{aligned}
\left\| (1 - \chi_\Omega) \widehat{f}_j \right\|_2^2 &= \int_{\mathbb{R}^d \setminus \Omega} \left| \widehat{f}_j(\boldsymbol{\omega}) \right|^2 d\boldsymbol{\omega} \\
&= |\det(\mathbf{D})|^{-2j} \int_{\mathbb{R}^d \setminus \Omega} \left| \widehat{f}((\mathbf{D}^T)^{-j} \boldsymbol{\omega}) \right|^2 \frac{1 + \left| \widehat{L}((\mathbf{D}^T)^{-j} \boldsymbol{\omega}) \right|^2}{1 + \left| \widehat{L}((\mathbf{D}^T)^{-j} \boldsymbol{\omega}) \right|^2} d\boldsymbol{\omega}
\end{aligned}$$

and since L is spline-admissible of order r

$$\left\| (1 - \chi_\Omega) \widehat{f}_j \right\|_2^2 \leq |\det(\mathbf{D})|^{2jr/d-j} \|f\|_{W_2^L}^2,$$

so that

$$|\det(\mathbf{D})|^{j/2} \left\| (1 - \chi_\Omega) \widehat{f}_j \right\|_2 \leq |\det(\mathbf{D})|^{jr/d} \|f\|_{W_2^L}. \quad (4)$$

In order to bound the remaining term, we will need a formula for the projection of $\widehat{f_j}\chi_\Omega$ onto V_j^j . Notice that

$$\begin{aligned} 1 - m_j((\mathbf{D}^T)^{-j}\cdot) &= 1 - \frac{|\widehat{\varphi_j} \circ (\mathbf{D}^T)^{-j}|^2}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi_j} \circ (\mathbf{D}^T)^{-j}(\cdot - 2\pi\mathbf{k})|^2} \\ &= 1 - \frac{|\widehat{\varphi_j \circ \mathbf{D}^j}|^2}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi_j \circ \mathbf{D}^j}(\cdot - 2\pi\mathbf{k})|^2}, \end{aligned}$$

so we can apply [8, Theorem 2.20] to get

$$\begin{aligned} E(\widehat{f_j}\chi_\Omega, \widehat{V_j^j})^2 &= \int_\Omega |\widehat{f_j}|^2 (1 - m_j((\mathbf{D}^T)^{-j}\cdot)) \\ &= |\det(\mathbf{D})|^{-2j} \int_\Omega |\widehat{f} \circ (\mathbf{D}^T)^{-j}|^2 (1 - m_j((\mathbf{D}^T)^{-j}\cdot)). \end{aligned}$$

Now, changing variables gives

$$\begin{aligned} E(\widehat{f_j}\chi_\Omega, \widehat{V_j^j})^2 &= |\det(\mathbf{D})|^{-j} \int_{(\mathbf{D}^T)^{-j}\Omega} |\widehat{f}|^2 (1 + |\widehat{L}|^2) \frac{1 - m_j}{1 + |\widehat{L}|^2} \\ &\leq |\det(\mathbf{D})|^{-j} \|f\|_{W_2^L}^2 \left\| \frac{1 - m_j}{1 + |\widehat{L}|^2} \right\|_{L_\infty((\mathbf{D}^T)^{-j}\Omega)}. \end{aligned}$$

Applying our assumption on $(1 - m_j)$, we have

$$|\det(\mathbf{D})|^{j/2} E(\widehat{f_j}\chi_\Omega, \widehat{V_j^j}) \leq c |\det(\mathbf{D})|^{jr/d} \|f\|_{W_2^L}. \quad (5)$$

Substituting the estimates (4) and (5) into (3) yields the result. \square

4 Operator-Like Wavelets and Riesz Bases

Using the non-stationary MRA defined in the previous section, we define the scale of wavelet spaces W_j by the relationship

$$V_j = V_{j+1} \oplus W_{j+1};$$

i.e., W_{j+1} is the orthogonal complement of V_{j+1} in V_j . Our goal in this section is to define Riesz bases for these spaces and for $L_2(\mathbb{R}^d)$. To begin, let us define the functions

$$\psi_{j+1} = L^* \phi_j,$$

which we claim generate Riesz bases for the wavelet spaces, under mild conditions on the operator L . First, note that ψ_{j+1} is indeed in V_j , because its Fourier transform $\widehat{\psi}_{j+1}$ is a periodic multiple of $\widehat{\phi}_j$, and thus

$$\widehat{\psi}_{j+1}(\omega) = |\det(\mathbf{D})|^j \frac{\widehat{L}_{d,j}(\omega)^*}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\phi}_j(\omega + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2} \widehat{\phi}_j(\omega).$$

A direct implication of our wavelet construction is the following property.

Property 4.1. The wavelet function ψ_{j+1} behaves like a multiscale version of the underlying operator L in the sense that, for any $f \in W_2^L$, we have $f * \psi_{j+1} = L^*\{f * \phi_j\}$. Hence, in the case where ϕ_j is a lowpass filter, $\{L^*\{f * \phi_j\}\}_{j \in \mathbb{Z}}$ corresponds to a multiscale representation of $L^*\{f\}$.

The next few results focus on showing that the $\mathbf{D}^j \mathbb{Z}^d / \mathbf{D}^{j+1} \mathbb{Z}^d$ shifts of $\widehat{\psi}_{j+1}$ are orthogonal to V_{j+1} and generate a Riesz basis of W_{j+1} .

Proposition 4.2. *The wavelets $\{\psi_{j+1}(\cdot - \mathbf{D}^j \mathbf{k})\}_{\mathbf{k} \in \mathbb{Z}^d \setminus \mathbf{D} \mathbb{Z}^d}$ are orthogonal to the space V_{j+1} .*

Proof. Note that it suffices to show $\langle \varphi_{j+1}, \psi_{j+1}(\cdot - \mathbf{D}^j \mathbf{k}) \rangle = 0$ for every $\mathbf{k} \in \mathbb{Z}^d \setminus \mathbf{D} \mathbb{Z}^d$. By definition

$$\begin{aligned} \langle \varphi_{j+1}, \psi_{j+1}(\cdot - \mathbf{D}^j \mathbf{k}) \rangle &= \int_{\mathbb{R}^d} \widehat{\varphi}_{j+1}(\omega) e^{i\omega \cdot \mathbf{D}^j \mathbf{k}} \widehat{L}(\omega) \widehat{\phi}_j(\omega) d\omega \\ &= \int_{\mathbb{R}^d} \widehat{L}_{d,j+1}(\omega) e^{i\omega \cdot \mathbf{D}^j \mathbf{k}} \widehat{\phi}_j(\omega) d\omega. \end{aligned}$$

Now let Ω be a fundamental domain for the lattice generated by $2\pi(\mathbf{D}^T)^{-j}$ so that we can write

$$\begin{aligned}
\langle \varphi_{j+1}, \psi_{j+1}(\cdot - \mathbf{D}^j \mathbf{k}) \rangle &= \int_{\Omega} \widehat{L}_{d,j+1}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega} \cdot \mathbf{D}^j \mathbf{k}} \sum_{\mathbf{n} \in \mathbb{Z}^d} \widehat{\phi}_j(\boldsymbol{\omega} - 2\pi(\mathbf{D}^T)^{-j} \mathbf{n}) d\boldsymbol{\omega} \\
&= \int_{\Omega} \widehat{L}_{d,j+1}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega} \cdot \mathbf{D}^j \mathbf{k}} d\boldsymbol{\omega} \\
&= 0.
\end{aligned}$$

The last equality holds because $\widehat{L}_{d,j+1}$ is periodic on a finer lattice. \square

In order to prove that shifts of ψ_j form a Riesz basis of the wavelet space, we need to introduce some notation, which will help formulate the problem as a shift-invariant one. Since we view the wavelet space as being shift-invariant on $\mathbf{D}^{j+1}\mathbb{Z}^d$, a representative wavelet is chosen for each of the $|\det(\mathbf{D})| - 1$ cosets. Specifically, recall that there are vectors $\{\mathbf{e}_l\}_{l=0}^{|\det(\mathbf{D})|-1}$ in \mathbb{Z}^d such that

$$\bigcup_l (\mathbf{D}^j \mathbf{e}_l + \mathbf{D}^{j+1} \mathbb{Z}^d) = \mathbf{D}^j \mathbb{Z}^d,$$

so for $1 \leq l \leq |\det(\mathbf{D})| - 1$, we define the wavelets

$$\psi_{j+1}^{(l)}(\mathbf{x}) := \psi_{j+1}(\mathbf{x} - \mathbf{D}^j \mathbf{e}_l).$$

In the following, necessary and sufficient conditions on the operator L will be given which guarantee that $\Psi = \{\psi_{j+1}^{(l)}\}_{l=1}^{|\det(\mathbf{D})|-1}$ generates a Riesz basis of W_{j+1} . The technique used is called fiberization and it can be applied to characterize finitely generated shift-invariant spaces [16]. In this setting, a collection of functions defines a Gramian matrix, and the property of being a Riesz basis is equivalent to the Gramian having bounded eigenvalues. In our situation, the Gramian for Ψ is

$$G_{\Psi}(\boldsymbol{\omega}) = \left(\sum_{\boldsymbol{\alpha} \in 2\pi(\mathbf{D}^T)^{-j-1}\mathbb{Z}^d} e^{-i\mathbf{D}^j(\mathbf{e}_k - \mathbf{e}_l) \cdot (\boldsymbol{\omega} + \boldsymbol{\alpha})} \left| \widehat{\psi}_{j+1}(\boldsymbol{\omega} + \boldsymbol{\alpha}) \right|^2 \right)_{k,l}.$$

Let us denote the largest and smallest eigenvalues of G_{Ψ} by $\Lambda(\boldsymbol{\omega})$ and $\lambda(\boldsymbol{\omega})$, respectively. Then the collection Ψ is a Riesz basis if and only if Λ and $1/\lambda$ are essentially bounded (cf. [16] Theorem 2.3.6). To simplify this

matrix without changing the eigenvalues, we apply the similarity transformation $T(\boldsymbol{\omega})^{-1}G_\Psi(\boldsymbol{\omega})T(\boldsymbol{\omega})$, where T is the diagonal matrix with diagonal entry $e^{-i\mathbf{D}^j \mathbf{e}_l \cdot \boldsymbol{\omega}}$ in row l . This transformation multiplies column l of G_Ψ by $e^{-i\mathbf{D}^j \mathbf{e}_l \cdot \boldsymbol{\omega}}$ and row k of G_Ψ by $e^{i\mathbf{D}^j \mathbf{e}_k \cdot \boldsymbol{\omega}}$. Since the eigenvalues are unchanged, let us call this new matrix G_Ψ as well. After applying the transformation, we have

$$G_\Psi(\boldsymbol{\omega}) = \left(\sum_{\boldsymbol{\alpha} \in 2\pi(\mathbf{D}^T)^{-j-1}\mathbb{Z}^d} e^{-i\mathbf{D}^j(\mathbf{e}_k - \mathbf{e}_l) \cdot \boldsymbol{\alpha}} \left| \widehat{\psi}_{j+1}(\boldsymbol{\omega} + \boldsymbol{\alpha}) \right|^2 \right)_{k,l}.$$

Using the fact that $\bigcup_m (\mathbf{e}_m + \mathbf{D}^T \mathbb{Z}^d) = \mathbb{Z}^d$ and the notation

$$c(m; \boldsymbol{\omega}) := \sum_{\boldsymbol{\beta} \in 2\pi(\mathbf{D}^T)^{-j}\mathbb{Z}^d} \left| \widehat{\psi}_{j+1}(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j-1}\mathbf{e}_m + \boldsymbol{\beta}) \right|^2,$$

we can write

$$G_\Psi(\boldsymbol{\omega}) = \left(\sum_{m=0}^{|\det(\mathbf{D})|-1} c(m; \boldsymbol{\omega}) e^{-2\pi i(\mathbf{e}_k - \mathbf{e}_l) \cdot (\mathbf{D}^T)^{-1}\mathbf{e}_m} \right)_{k,l}.$$

Since G_Ψ is Hermitian, its maximum and minimum eigenvalues can be determined by analyzing the quadratic form

$$\begin{aligned} \boldsymbol{\alpha}^* G_\Psi \boldsymbol{\alpha} &= \sum_{k,l=1}^{|\det(\mathbf{D})|-1} \boldsymbol{\alpha}_k^* \boldsymbol{\alpha}_l \sum_{m=0}^{|\det(\mathbf{D})|-1} c(m; \boldsymbol{\omega}) e^{-2\pi i(\mathbf{e}_k - \mathbf{e}_l) \cdot (\mathbf{D}^T)^{-1}\mathbf{e}_m} \\ &= \sum_{m=0}^{|\det(\mathbf{D})|-1} c(m; \boldsymbol{\omega}) \sum_{k,l=1}^{|\det(\mathbf{D})|-1} (\boldsymbol{\alpha}_k e^{2\pi i \mathbf{e}_k \cdot (\mathbf{D}^T)^{-1}\mathbf{e}_m})^* \boldsymbol{\alpha}_l e^{2\pi i \mathbf{e}_l \cdot (\mathbf{D}^T)^{-1}\mathbf{e}_m} \\ &= \sum_{m=0}^{|\det(\mathbf{D})|-1} c(m; \boldsymbol{\omega}) \left| \sum_{k=1}^{|\det(\mathbf{D})|-1} \boldsymbol{\alpha}_k e^{2\pi i \mathbf{e}_k \cdot (\mathbf{D}^T)^{-1}\mathbf{e}_m} \right|^2 \end{aligned}$$

over all unit vectors $\boldsymbol{\alpha} \in \mathbb{C}^{|\det(\mathbf{D})|-1}$. Let us additionally define \mathbf{H} to be the inverse DFT matrix for the lattice generated by \mathbf{D} , i.e.,

$$\mathbf{H} = \left(e^{2\pi i e_k \cdot (\mathbf{D}^T)^{-1} e_m} \right)_{k,m}$$

with k and m ranging from 0 to $|\det(\mathbf{D})| - 1$, and let \mathbf{H}_0 be the submatrix obtained by removing row 0 from \mathbf{H} . Note that each sum in the quadratic form above is given by the inner product of $\boldsymbol{\alpha}$ with a column of \mathbf{H}_0 . This identification allows us to prove the following.

Lemma 4.3. *For a fixed $\boldsymbol{\omega}$, the quadratic form $\boldsymbol{\alpha}^* G_{\Psi} \boldsymbol{\alpha}$ is non-zero if and only if at most one of the coefficients $c(m; \boldsymbol{\omega})$ is zero.*

Proof. First, suppose that two distinct coefficients, $c(m_0; \boldsymbol{\omega})$ and $c(m_1; \boldsymbol{\omega})$, are zero. Then we can clearly find an $\boldsymbol{\alpha}$ that is orthogonal to the remaining $|\det(\mathbf{D})| - 2$ columns of \mathbf{H}_0 . Hence the quadratic form will be 0 for such an $\boldsymbol{\alpha}$.

Next, suppose that all coefficients except possibly $c(m_0; \boldsymbol{\omega})$ are non-zero, and denote the matrix obtained from \mathbf{H}_0 by removing column m_0 by \mathbf{H}_{0,m_0} . Then, to prove positivity of the quadratic form, we show that the columns of \mathbf{H}_{0,m_0} span $\mathbb{C}^{|\det(\mathbf{D})|-1}$. This is equivalent to showing that the determinant of \mathbf{H}_{0,m_0} is non-zero, which follows from the fact that \mathbf{H}^{-1} has non-zero entries (since \mathbf{H}_{0,m_0} is a first minor of \mathbf{H}). \square

Lemma 4.4. *The collection Ψ is a Riesz basis if and only if no two of the coefficients $c(m; \boldsymbol{\omega})$ are zero for the same $\boldsymbol{\omega}$.*

Proof. This follows from the definition of ψ_{j+1} and Lemma 4.3. \square

Suppose that the conditions of Theorem 4.5 are satisfied so that Ψ is a Riesz basis. Then the orthogonality between the shifts of Ψ and V_{j+1} , together with the Riesz basis condition on φ_{j+1} , implies $\Psi' := \{\varphi_{j+1}\} \cup \Psi$ generates a Riesz basis. The fact that Ψ' spans V_j follows from a comparison with the Riesz basis generated by the $\mathbf{D}^{j+1} \mathbb{Z}^d$ translates of $\varphi_j(\cdot - \mathbf{D}^j \mathbf{e}_l)$, cf. [1] and [7, Theorem 2.26]. As a consequence of these results, Ψ would be a stable basis of W_{j+1} .

In order to interpret this result in terms of the operator L , we note that $\widehat{\psi}_{j+1}(\boldsymbol{\omega})$ vanishes at $\mathbf{p} + (\mathbf{D}^T)^{-j} 2\pi \mathbf{k}$ for every $\mathbf{p} \in \mathbb{R}^d$ such that $\widehat{L}(\mathbf{p}) = 0$. In other words, each zero of \widehat{L} generates a periodic sequence of zeros in the spectrum of the generating wavelet. As long as these periodizations of zeros are essentially disjoint, the conditions of Theorem 4.5 will be satisfied. The

zero set of the symbol is denoted by $\mathcal{N} = \{\mathbf{p} \in \mathbb{R}^d | \widehat{L}(\mathbf{p}) = 0\}$, and for each scale j , we define

$$\mathcal{N}_j^{(l)} = \mathbf{p} + (\mathbf{D}^T)^{-(j+1)} 2\pi \mathbf{e}_l + (\mathbf{D}^T)^{-j} 2\pi \mathbb{Z}^d, \quad l = 0, \dots, |\det(\mathbf{D})| - 1,$$

where $p \in \mathcal{N}$.

Theorem 4.5. *Let $j \in \mathbb{Z}$ be an arbitrary scale. Then the family of functions $\Psi = \{\psi_{j+1}^{(l)}\}_{l=1}^{|\det(\mathbf{D})|-1}$ generates a Riesz basis of W_{j+1} if and only if the sets $\mathcal{N}_j^{(l)}$ satisfy*

$$\mathcal{N}_j^{(0)} \cap \mathcal{N}_j^{(l)} = \emptyset \quad (6)$$

for each $1 \leq l \leq |\det(\mathbf{D})| - 1$.

Proof. This follows from the periodicity of the zeros of $\widehat{\psi}_{j+1}$ and the condition on the coefficients from Lemma 4.4. \square

Our wavelet construction is intended to be general so that we may account for a large collection of operators. As a consequence of this generality, we can not conclude that our wavelet construction always produces a Riesz basis of $L_2(\mathbb{R}^d)$. Here, we shall impose additional conditions to ensure a Riesz basis is produced. In order to preserve generality, we shall focus on the fine scale wavelet spaces and include a B-spline space V_{j_0} in our Riesz basis.

Theorem 4.6. *Let L be a spline admissible operator of order r , and suppose that there exist $\omega_0 > 0$ and constants $C_1, C_2 > 0$ such that the symbol \widehat{L} satisfies*

$$C_1 |\boldsymbol{\omega}|^r \leq \left| \widehat{L}(\boldsymbol{\omega}) \right| \leq C_2 |\boldsymbol{\omega}|^r$$

for $|\boldsymbol{\omega}| \geq \omega_0$. Then there is an integer j_0 such that the collection

$$\{\varphi_{j_0+1}(\cdot - \beta)\}_{\beta \in \mathbf{D}^{j_0} \mathbb{Z}^d} \bigcup_{j \leq j_0} \{|\det(\mathbf{D})|^{(j+1)/(2d)-j} \psi_{j+1}(\cdot - \beta)\}_{\beta \in \mathbf{D}^j \mathbb{Z}^d \setminus \mathbf{D}^{j+1} \mathbb{Z}^d} \quad (7)$$

forms a Riesz basis of $L_2(\mathbb{R}^d)$.

Proof. We shall verify the result for the dilation matrix $\mathbf{D} = 2\mathbf{I}$, as the other cases are similar. Let j_0 be an integer for which $\omega_0 < 2\pi 2^{-j_0-2}$.

Considering Lemma 4.4, we can see that the Riesz bounds for the wavelet spaces depend on estimating the functions $c(m; \boldsymbol{\omega})$, the main term being the sum

$$\sum_{\boldsymbol{\beta} \in 2\pi 2^{-j} \mathbb{Z}^d} \left| \frac{\widehat{L}(\boldsymbol{\omega} + 2\pi 2^{-j-1} \mathbf{e}_m + \boldsymbol{\beta})^{-1}}{\sum_{\mathbf{k} \in \mathbb{Z}^d} \left| \widehat{L}(\boldsymbol{\omega} + 2\pi 2^{-j-1} \mathbf{e}_m + \boldsymbol{\beta} + 2\pi 2^{-j} \mathbf{k}) \right|^{-2}} \right|^2. \quad (8)$$

The value of this sum depends on the position of the lattice

$$X_j(m, \boldsymbol{\omega}) := \boldsymbol{\omega} + 2\pi 2^{-j-1} \mathbf{e}_m + 2\pi 2^{-j} \mathbb{Z}^d$$

with respect to the origin. To prove bounds of (8), let us introduce the notation h_j for the fill distance and q_j for the separation radius of the lattices $X_j(m, \boldsymbol{\omega})$; i.e.,

$$h_j := \sup_{\mathbf{x} \in \mathbb{R}^d} \inf_{\mathbf{x}_k \in X_j(0,0)} |\mathbf{x} - \mathbf{x}_k| = 2\pi 2^{-j-1} \sqrt{d}$$

and

$$q_j := \inf_{k \neq k'} |\mathbf{x}_k - \mathbf{x}_{k'}| = 2\pi 2^{-j-1}.$$

Since each lattice $X_j(m, \boldsymbol{\omega})$ is a translation of $X_j(0, 0)$, the quantities h_j and q_j are independent of m and $\boldsymbol{\omega}$.

We shall now bound (8) by considering two cases:

1. $\text{dist}(0, X_j(m, \boldsymbol{\omega})) \geq q_j/2$;
2. $\text{dist}(0, X_j(m, \boldsymbol{\omega})) < q_j/2$.

In the first case, all points of the lattice $X_j(m, \boldsymbol{\omega})$ lie outside of the ball of radius ω_0 centered at the origin. Therefore, (8) can be reduced to

$$\left(\sum_{\mathbf{x}_k \in X_j(m, \boldsymbol{\omega})} \left| \widehat{L}(\mathbf{x}_k) \right|^{-2} \right)^{-1}. \quad (9)$$

Applying Proposition A.1 and Proposition A.2, we can find upper and lower bounds for this sum which are proportional to 2^{-j-1} . Importantly, the proportionality constants are independent of j .

For the second case, we must be more careful as one of the lattice points will lie close to the origin. For any fixed ω , there will be at most one m for which the lattice $X_j(m, \omega)$ satisfies this second condition. Therefore, a sufficient lower bound for the lattice sum will be 0; however, the upper bound must match the one derived above. Let us consider the cases

2a. \widehat{L} takes the value 0 on the lattice $X_j(m, \omega)$

2b. $|\widehat{L}^{-1}| < \infty$ on the lattice $X_j(m, \omega)$

If the first condition is satisfied, then (8) will be 0. However, if the second condition holds, then we can again reduce the sum to (9). Clearly, 0 is a lower bound for this expression. For the upper bound, we apply Proposition A.1 as in the previous case.

To finish the proof, we note that Lemma 4.3 implies that the wavelets

$$\{\psi_{j+1}(\cdot - \mathbf{D}^j \mathbf{k})\}_{\mathbf{k} \in \mathbb{Z}^d \setminus \mathbf{D}\mathbb{Z}^d}$$

form a Riesz basis at each level $j \leq j_0$. Furthermore, the bounds on $c(m, \omega)$ imply that the Riesz bounds are all equal. Therefore, the collection (7) is a Riesz basis of $L_2(\mathbb{R}^d)$. \square

5 Decorrelation of Coefficients

As was stated in the introduction, the primary reason for our construction is to promote a sparse wavelet representation. Our model is based on the assumption that the wavelet coefficients of a signal s are computed by the L_2 inner product

$$\langle \psi_j(\cdot - \mathbf{D}^j \mathbf{k}), s \rangle.$$

Here, we should point out that, unless the wavelets form an orthogonal basis, reconstruction will be defined in terms of a dual basis. However, as our focus in this paper is the sparsity of the coefficients, we shall be content to work with the analysis component of the approximation and leave the synthesis component for future study.

Now, considering our stochastic model, it is important to use wavelets that (nearly) decorrelate the signal within each scale, and one way to accomplish this goal is by modifying the underlying operator. Hence, given a spline-admissible pair (L, \mathbf{D}) , we define a new spline-admissible pair (L_n, \mathbf{D}) by $\widehat{L}_n := \widehat{L}^n$, and we will see that as n increases, the wavelet coefficients become decorrelated. This result follows from the fact that the $(L_n)^*L_n$ -spline interpolants (appropriately scaled) converge to a sinc-type function, and it is motivated by the work of Aldroubi and Unser, which shows that a broad family of spline-like interpolators converge to the ideal sinc interpolator [2]. To state this result explicitly, we denote the B-splines for L_n by $\widehat{\varphi}_{n,j} = \widehat{\varphi}_j^n$. Therefore the $(L_n)^*L_n$ -spline interpolants are given by

$$\begin{aligned}\widehat{\phi}_{n,j}(\boldsymbol{\omega}) &= |\det(\mathbf{D})|^j \frac{|\widehat{\varphi}_{n,j}(\boldsymbol{\omega})|^2}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_{n,j}(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2} \\ &= |\det(\mathbf{D})|^j \frac{|\widehat{\varphi}_j(\boldsymbol{\omega})|^{2n}}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^{2n}},\end{aligned}$$

and we analogously define

$$m_{n,j}(\boldsymbol{\omega}) = \frac{|\widehat{\varphi}_j(\boldsymbol{\omega})|^{2n}}{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_j(\boldsymbol{\omega} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^{2n}}.$$

Decorrelation will depend on showing that the functions $m_{n,j}$ converge to 1 almost everywhere on some fundamental domain Ω_j of $2\pi(\mathbf{D}^T)^{-j}\mathbb{Z}^d$ and to 0 almost everywhere outside of Ω_j . Our proof relies on techniques used in the analysis of radial basis functions. For example, a similar method was used by Baxter to prove convergence of the Lagrange functions associated with multiquadric functions [3, Chapter 7]. The idea is to define disjoint sets covering \mathbb{R}^d , each with a single point in any given fundamental domain, and analyze the convergence of $m_{n,j}$ on these sets.

Definition 5.1. For each $j \in \mathbb{Z}$ and $\mathbf{x} \in \Omega_j$ we define the set

$$E_{j,\mathbf{x}} := \{\mathbf{x} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k} : \mathbf{k} \in \mathbb{Z}^d\}.$$

By the Riesz basis condition, each $E_{j,\mathbf{x}}$ has a finite number of elements y with $m_{1,j}(\mathbf{y})$ of maximal size, and we define F_j to be the set of $\mathbf{x} \in \Omega_j$ such that there is not a unique $y \in E_{j,\mathbf{x}}$ where $m_{1,j}$ attains a maximum.

Lemma 5.2. *Let $\mathbf{x} \in \Omega_j \setminus F_j$, then for $\mathbf{y} \in E_{j,\mathbf{x}}$ we have $m_{n,j}(\mathbf{y}) \rightarrow 0$ if and only if $m_{1,j}(\mathbf{y})$ is not of maximal size over $E_{j,\mathbf{x}}$. Furthermore, if $m_{1,j}(\mathbf{y})$ is of maximal size, then $m_{n,j}(\mathbf{y}) \rightarrow 1$.*

Proof. Fix $\mathbf{x} \in \Omega_j \setminus F_j$ and $\mathbf{y} \in E_{j,\mathbf{x}}$. Notice that the periodicity of the denominator of $m_{1,j}$ implies that $m_{1,j}(\mathbf{y})$ is maximal iff $|\widehat{\varphi}_{1,j}(\mathbf{y})|$ is maximal.

Let us first suppose $m_{1,j}(\mathbf{y})$ is not maximal. If $|\widehat{\varphi}_{1,j}(\mathbf{y})| = 0$, the result is obvious. Otherwise, there is some $\mathbf{k}_0 \in \mathbb{Z}^d$ and $b < 1$ such that

$$|\widehat{\varphi}_{1,j}(\mathbf{y})| \leq b |\widehat{\varphi}_{1,j}(\mathbf{y} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k}_0)|.$$

Therefore

$$\begin{aligned} |\widehat{\varphi}_{n,j}(\mathbf{y})|^2 &\leq b^{2n} |\widehat{\varphi}_{n,j}(\mathbf{y} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k}_0)|^2 \\ &\leq b^{2n} \sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{\varphi}_{n,j}(\mathbf{y} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})|^2 \end{aligned}$$

and the result follows.

Next, suppose $m_{1,j}(\mathbf{y})$ is of maximal size. Since $\widehat{\varphi}_{1,j}$ has no periodic zeros, $|\widehat{\varphi}_{1,j}(\mathbf{y})| \neq 0$. Therefore

$$m_{n,j}(\mathbf{y}) = \frac{1}{B_{n,j}(\mathbf{y})}$$

with

$$B_{n,j}(\mathbf{y}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \left| \frac{\widehat{\varphi}_{1,j}(\mathbf{y} + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})}{\widehat{\varphi}_{1,j}(\mathbf{y})} \right|^{2n}.$$

Since $|\widehat{\varphi}_{1,j}(\mathbf{y})|$ is of maximal size, all terms of the sum except one are less than 1. In particular, $B_{n,j}$ will converge to 1 as n increases. \square

Lemma 5.3. *Given $j \in \mathbb{Z}$, if the Lebesgue measure of F_j is 0, then*

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} m_{n,j}(\cdot + 2\pi(\mathbf{D}^T)^{-j}\mathbf{k})^2$$

converges to 1 in $L_1(\Omega_j)$ as $n \rightarrow \infty$.

Proof. Notice that the sum is bounded by 1. Therefore Lemma 5.2 implies that the sum converges to 1 on the complement of F_j . Hence, we can apply the dominated convergence theorem to obtain the result. \square

With this theorem, we can show how the wavelets corresponding to L_n decorrelate within scale as n becomes large. The way we characterize decorrelation is in terms of the semi-inner products

$$(f, g)_n := \int_{\mathbb{R}^d} \widehat{f} \widehat{g}^* \left| \widehat{L}_n \right|^{-2},$$

which is a true inner product for the wavelets

$$\psi_{n,j+1} = L_n^* \{\phi_{n,j}\}.$$

Theorem 5.4. *Suppose the Lebesgue measure of $\cup_{j \in \mathbb{Z}} F_j$, as provided in Definition 5.1, is 0. Then as n increases, the wavelet coefficients decorrelate in the following sense. For any $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$ we have*

$$(\psi_{n,j+1}, \psi_{n,j+1}(\cdot - \mathbf{D}^j \mathbf{k}))_n \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. The inner product can be expressed as an integral using the definition

$$\begin{aligned} (\psi_{n,j+1}, \psi_{n,j+1}(\cdot - \mathbf{D}^j \mathbf{k}))_n &= \int_{\mathbb{R}^d} \widehat{\psi}_{n,j+1}(\boldsymbol{\omega}) (\psi_{n,j+1}(\cdot - \mathbf{D}^j \mathbf{k}))^\wedge(\boldsymbol{\omega})^* \left| \widehat{L}_n \right|^{-2} d\boldsymbol{\omega} \\ &= |\det(\mathbf{D})|^{2j} \int_{\mathbb{R}^d} m_{n,j}(\boldsymbol{\omega})^2 e^{-i\mathbf{D}^j \mathbf{k} \cdot \boldsymbol{\omega}} d\boldsymbol{\omega}, \end{aligned}$$

and we can periodize the integrand to get

$$\begin{aligned} \int_{\mathbb{R}^d} m_{n,j}(\boldsymbol{\omega})^2 e^{-i\mathbf{D}^j \mathbf{k} \cdot \boldsymbol{\omega}} d\boldsymbol{\omega} &= \sum_{\mathbf{l} \in 2\pi(\mathbf{D}^T)^{-j} \mathbb{Z}^d} \int_{\Omega_j + \mathbf{l}} m_{n,j}(\boldsymbol{\omega})^2 e^{-i\mathbf{D}^j \mathbf{k} \cdot \boldsymbol{\omega}} d\boldsymbol{\omega} \\ &= \int_{\Omega_j} e^{-i\mathbf{D}^j \mathbf{k} \cdot \boldsymbol{\omega}} \sum_{\mathbf{l} \in 2\pi(\mathbf{D}^T)^{-j} \mathbb{Z}^d} m_{n,j}(\boldsymbol{\omega} - \mathbf{l})^2 d\boldsymbol{\omega}. \end{aligned}$$

The last expression converges to 0 by the Lebesgue dominated convergence theorem. \square

Let us now show how this result implies decorrelation of the wavelet coefficients. To begin, we introduce the notation

$$c_{n,j+1,\mathbf{k}} := \langle \psi_{n,j}(\cdot - \mathbf{D}^j \mathbf{k}), s \rangle$$

for the wavelet coefficients of the random signal s . Note that our stochastic model, $Ls = w$, implies that the coefficients are random variables. Hence, for distinct \mathbf{k} and \mathbf{k}' , the covariance between $c_{n,j+1,\mathbf{k}}$ and $c_{n,j+1,\mathbf{k}'}$ is determined by the expected value of their product:

$$\mathbb{E}\{c_{n,j+1,\mathbf{k}}c_{n,j+1,\mathbf{k}'}\} = \mathbb{E}\{\langle \phi_{n,j}(\cdot - \mathbf{D}^j \mathbf{k}), w \rangle \langle \phi_{n,j}(\cdot - \mathbf{D}^j \mathbf{k}'), w \rangle\}.$$

Now, as long as the white noise w has zero mean and second-order moments, the covariance satisfies

$$\begin{aligned} \mathbb{E}\{c_{n,j+1,\mathbf{k}}c_{n,j+1,\mathbf{k}'}\} &= \langle \phi_{n,j}(\cdot - \mathbf{D}^j \mathbf{k}), \phi_{n,j}(\cdot - \mathbf{D}^j \mathbf{k}') \rangle \\ &= (\psi_{n,j+1}, \psi_{n,j+1}(\cdot - \mathbf{D}^{j+1} \mathbf{k}))_n \\ &\rightarrow 0, \end{aligned}$$

where the convergence follows from Theorem 5.4. Therefore, when the standard deviations of $c_{n,j+1,\mathbf{k}}$ and $c_{n,j+1,\mathbf{k}'}$ are bounded below, the correlation between them will also converge to zero.

6 Examples

The wavelet construction that we have presented in this paper is quite general and accommodates many operators. In this section, we consider operators that are defined as polynomials in the Laplacian. In particular, we shall show how spline admissibility may be verified for such operators. Also, note that each of these operators satisfies the growth condition of Theorem 4.6. Therefore, the corresponding wavelet spaces may be used to construct Riesz bases of $L_2(\mathbb{R}^d)$.

6.1 Matérn and Laplace Operators

First, consider the two-dimensional Matérn operator that is defined by the symbol $\widehat{L}_\nu(\boldsymbol{\omega}) = (|\boldsymbol{\omega}|^2 + 1)^{\nu/2}$, with the parameter $\nu > 1$. As $\widehat{L}_\nu(\boldsymbol{\omega})^{-1}$ satisfies the Riesz basis condition, no localization operator is needed. Therefore, the operator L_ν is spline-admissible of order ν . Also note that the fractional Laplacian $\widehat{L} = |\boldsymbol{\omega}|^3$ and iterated Laplacian $\widehat{L} = |\boldsymbol{\omega}|^4$ operators are also spline-admissible.

Each of these wavelets can be considered with any admissible subsampling matrix \mathbf{D} . In two dimensions, the choice of $\mathbf{D} = 2\mathbf{I}$ corresponds to the rectangular grid, while

$$\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

characterizes the quincunx subsampling scheme.

6.2 Helmholtz Operator

Operators whose symbols have more general zero sets can also be used. As an example, consider the polynomial $p(x) = 1/4 + x$. The operator $p(\Delta)$ is known as the Helmholtz operator, and its symbol is $\widehat{L}(\boldsymbol{\omega}) = 1/4 - |\boldsymbol{\omega}|^2$. The wavelets corresponding to this operator could potentially have applications in optics, as the Helmholtz equation,

$$\Delta u + \lambda u = f,$$

is a reduced form of the wave equation [9, Chapter 5]. In what follows, we show that this is a spline-admissible operator for the scaling matrix $\mathbf{D} = 2\mathbf{I}$ on \mathbb{R}^2 . However, since the wavelets $\psi_j = L^* \phi_j$ will not form a Riesz basis for the coarse-scale wavelet spaces, we only consider $j \leq -1$.

The primary issue with proving spline-admissibility is defining discrete operators $L_{d,j}$ corresponding to L . It is known that sufficiently smooth functions have absolutely convergent Fourier series [10, Theorem 3.2.9], so we must construct such a function with the same zero set as \widehat{L} . Since \widehat{L} is zero on a circle and radial, we can choose $L_{d,j}$ to be the periodization of a radial function. More precisely, we define f_j to be a function that is periodic with respect to the lattice dual to $2^j \mathbb{Z}^2$ and, on $[-2\pi, 2\pi]^2$, satisfies:

1. The function $f_j(\boldsymbol{\omega})$ has continuous derivatives up to order 3 and is constant for $|\boldsymbol{\omega}| \geq 1$;
2. There are constants C_1 and C_2 such that

$$0 < C_1 \leq f_j(\boldsymbol{\omega})(1/4 - |\boldsymbol{\omega}|^2) < C_2 < \infty$$

for $|\boldsymbol{\omega}| \leq 1$.

These conditions are met, for example, by defining f_j by

$$f_j(\boldsymbol{\omega}) = \begin{cases} \phi_{2,k}(|\boldsymbol{\omega}|) - \phi_{2,k}(1/2) & \text{if } |\boldsymbol{\omega}| < 1 \\ -\phi_{2,k}(1/2) & \text{if } |\boldsymbol{\omega}| \geq 1, \end{cases}$$

where $\phi_{2,k}(|\cdot|)$ is a Wendland function with $2k$ continuous derivatives [23, Section 9.4]. We can therefore apply our wavelet construction, which produces a Riesz basis.

A Discrete Sums

Let $X = \{\mathbf{x}_k\}$ be a countable collection of points in \mathbb{R}^d , and define

$$h_X := \sup_{\mathbf{x} \in \mathbb{R}^d} \inf_{\mathbf{x}_k \in X} |\mathbf{x} - \mathbf{x}_k|$$

$$q_X := \inf_{k \neq k'} |\mathbf{x}_k - \mathbf{x}_{k'}|.$$

Also, let $B(\mathbf{x}, r)$ denote the ball of radius r centered at \mathbf{x} . Proving Riesz bounds for the wavelet spaces relies on the following propositions concerning sums of function values over discrete sets.

Proposition A.1. *If $h_X < \infty$ and $r > d/2$, then there exists a constant $C > 0$ (depending only on r and d , not h_X) such that*

$$\sum_{|\mathbf{x}_k| > 2h_X} |\mathbf{x}_k|^{-2r} \geq Ch_X^{-2r}$$

Proof. Since $\mathbf{x}_k \geq 2h_X$, we have

$$\left| \mathbf{x}_k - h_X \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right| \geq 2^{-1} |\mathbf{x}_k|,$$

which implies

$$|\mathbf{x}_k|^{-2r} \geq 2^{-2r} \left| \mathbf{x}_k - h_X \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right|^{-2r}.$$

Then

$$\begin{aligned} \sum_k |\mathbf{x}_k|^{-2r} &\geq \sum_{|\mathbf{x}_k| \geq 2h_X} |\mathbf{x}_k|^{-2r} \\ &\geq 2^{-2r} \sum_{|\mathbf{x}_k| \geq 2h_X} \left| \mathbf{x}_k - h_X \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right|^{-2r} \frac{\text{Vol}(B(\mathbf{x}_k, h_X))}{\text{Vol}(B(\mathbf{x}_k, h_X))} \\ &\geq \frac{2^{-2r}}{\text{Vol}(B(0, h_X))} \sum_{|\mathbf{x}_k| \geq 2h_X} \int_{B(\mathbf{x}_k, h_X)} |\mathbf{x}|^{-2r} d\mathbf{x} \\ &\geq Ch_X^d \int_{3h_X}^{\infty} t^{-2r+(d-1)} dt \\ &\geq Ch_X^{-2r} \end{aligned}$$

□

Proposition A.2. *If $|\mathbf{x}_k| > q_X/4$ and $r > d/2$, then there exists a constant $C > 0$ (depending only on r and d , not q_X) such that*

$$\sum_k |\mathbf{x}_k|^{-2r} \leq Cq_X^{-2r}$$

Proof. Using the fact that $|\mathbf{x}_k| > q_X/4$, we can write

$$\left| \mathbf{x}_k + \frac{q_X}{8} \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right| \leq 2|\mathbf{x}_k|,$$

which implies

$$|\mathbf{x}_k|^{-2r} \leq 2^{2r} \left| \mathbf{x}_k + \frac{q_X}{8} \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right|^{-2r}.$$

We now have

$$\begin{aligned}
\sum_k |\mathbf{x}_k|^{-2r} &\leq 2^{2r} \sum_k \left| \mathbf{x}_k + \frac{q_X}{8} \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \right|^{-2r} \frac{\text{Vol}(B(\mathbf{x}_k, q_X/8))}{\text{Vol}(B(\mathbf{x}_k, q_X/8))} \\
&\leq \frac{2^{2r}}{\text{Vol}(B(0, q_X/8))} \sum_k \int_{B(\mathbf{x}_k, q_X/8)} |\mathbf{x}|^{-2r} d\mathbf{x} \\
&\leq \frac{2^{2r}}{\text{Vol}(B(0, q_X/8))} \int_{|\mathbf{x}| > q_X/8} |\mathbf{x}|^{-2r} d\mathbf{x} \\
&\leq C q_X^{-d} \int_{q_X/8}^{\infty} t^{-2r+(d-1)} dt \\
&\leq C q_X^{-2r}.
\end{aligned}$$

□

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